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LETTER TO THE EDITOR

Iterative solution of nonlinear partial differential equations

Richard Blender

Institut für Meteorologie, Freie Universität Berlin, Thielallee 50, D-1000 Berlin 33, Federal Republic of Germany

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Abstract. We found that certain solutions of some nonlinear partial differential equations can be obtained easily by an iteration method. As examples, the Korteweg–de Vries equation (also the 'modified' version), the 1D and 2D Burgers equation and the Kadomtsev–Petviashvili equation have been studied. For the Liouville equation we found the general solution.

In this letter we describe an iteration method which in some cases is able to calculate exact solutions of nonlinear partial differential equations. In contrast to the successive approximation (Picard 1890) we look for the occurrence of the non-differentiated function u (typically in an advective term) and solve for it. The resulting differentiations can be performed efficiently by a computer. Clearly, we have limited capabilities of incorporating boundary conditions. At least we can use some free parameters which can be adjusted in the end if they are not eliminated during the iteration process. As initial guesses we try simple functions, e.g. waves.

The main point is that in some cases our method converges rapidly (after typically two steps) to an exact solution. For one example we could even find the general solution. Unfortunately, no convergence properties are known and the observed behaviour could be a mere accident. We will see that the method shows up as a generalization of trying an ansatz. The discussion of the physical origins and the applications will be restricted to a minimum (see Whitham 1974, Dodd 1982). We will also only briefly refer to the solution methods for the equations studied, e.g. spectral transform and Bäcklund transform (see Drazin 1983, Calogero 1988, Lamb 1980).

Let us begin with the well known Korteweg–de Vries (KdV) equation (Korteweg and de Vries 1895)

$$u_t - 6uu_x + u_{xxx} = 0. \tag{1}$$

Originally, this equation was used to describe surface water waves. Later on it was found that it can be applied to many problems in different areas. The initial value problem can be solved via the spectral transform. To apply our method we solve for u and define the iteration scheme

$$u^{(n+1)} = \tfrac{1}{6}[u_t^{(n)} + u_{xxx}^{(n)}]/u_x^{(n)} \qquad n \geq 0. \tag{2}$$

The simple guess $u^{(0)} = \log(x - ct)$, as well as the powers $(x - ct)^k$, converge within two steps to the moving singularity

$$u = -c/6 + 2(x - ct)^{-2}. \tag{3}$$

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To avoid that the term $u^{(0)}_{xxx}$ in (2) would vanish at the beginning, we had to use powers $k>3$ or $k<0$. Then we tried the wave ansatz $u^{(0)}=\cos[a(x-ct)]^{-k}$, e.g. $k=-1$. Within two steps we arrived at

$$u=4a^2/3-c/6+2a^2\tan^2[a(x-ct)]. \tag{4}$$

By demanding that $u\to 0$ for $|x|\to\infty$, which gives $a=\mathrm{i}\sqrt{c}/2$, (4) becomes the well known soliton

$$u=-\frac{c}{2}\operatorname{sech}^2\left[\frac{\sqrt{c}}{2}(x-ct)\right]. \tag{5}$$

More generally, the Jacobian elliptic functions (Abramowitz and Stegun 1970) sn, cn, dn(ϕ, m), $\phi=a(x-ct)$, all converge to $-2a^2m\,\mathrm{cn}^2(\phi, m)+\frac{2}{3}a^2(2m-1)-c/6$. Unfortunately, we could not obtain multi-soliton solutions by starting with simple combinations of waves.

Our next example is the *modified KdV equation* (Wadati 1973)

$$u_t+6u^2u_x+u_{xxx}=0 \tag{6}$$

which is also solvable by the spectral transform. This is interesting because of the occurrence of the square of u. We had difficulties to implement procedures which either simply divide by u or use the square root. However, we could get an equation with u linear by dividing (6) by u_x and differentiating with respect to x

$$\partial_x\left(\frac{u_t+u_{xxx}}{u_x}+6u^2\right)=0. \tag{7}$$

We have to keep in mind that due to the differentiation, we will solve (6) with an arbitrary time-dependent function $f(t)$

$$u_t+6u^2u_x+u_{xxx}=f(t)u_x. \tag{8}$$

We started with $u^{(0)}=b/\cos[a(x-ct)]$ and found an alternating behaviour of the iteration unless $a=\pm\mathrm{i}b$. On the other hand, the condition that $f(t)$ in (8) has to vanish yielded $c=b^2$, so finally (Zabusky 1967)

$$u=b\operatorname{sech}[b(x-b^2t)]. \tag{9}$$

Here we recognize that the common method of trying an ansatz is incorporated. If we had started with the correct functional form containing free parameters, these would have been determined already at the first step by the condition that we are at a fixed-point of the iteration.

Let us now turn to the *Burgers equation* (Burgers 1948) which served as an aid for the understanding of turbulence. It is derived from the Navier–Stokes equations by neglecting the pressure. The Burgers equation can be mapped onto the diffusion equation through the Hopf–Cole transformation (Hopf 1950). Firstly, we consider the 1D case

$$u_t+uu_x=u_{xx}. \tag{10}$$

The waves $u^{(0)}=\sin[a(x-ct)]$ lead to

$$u=c+2a\tan[a(x-ct)] \tag{11}$$

from which a kink-type solution can be constructed with a imaginary. Let us mention that $u^{(0)}=\log(x-ct)$ and $u^{(0)}=\tan[a(x/t-c)]$ reach fixed points as well.

A kink solution can also be found for the 2D Burgers equation

$$\begin{aligned} u_t + uu_x + vu_y - u_{xx} - u_{yy} &= 0 \\ v_t + uv_x + vv_y - v_{xx} - v_{yy} &= 0. \end{aligned} \tag{12}$$

We write these equations in matrix form and solve for (u, v)

$$\begin{pmatrix} u \\ v \end{pmatrix} = -\begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}^{-1} \begin{pmatrix} u_t - u_{xx} - u_{yy} \\ v_t - v_{xx} - v_{yy} \end{pmatrix}. \tag{13}$$

If we start with $u^{(0)} = a_1 \sinh(\phi_1)$ and $v^{(0)} = a_2 \sinh(\phi_2)$, $\phi_i = x + b_i y - f_i t$ we obtain

$$u = [2(1+b_1^2)b_2 \tanh(\phi_1) - 2(1+b_2^2)b_1 \tanh(\phi_2) + b_1 f_2 - b_2 f_1]/(b_1 - b_2)$$

$$v = [2(1+b_1^2)\tanh(\phi_1) - 2(1+b_2^2)\tanh(\phi_2) + f_2 - f_1]/(b_2 - b_1).$$

The fourth example is the *Kadomtsev–Petviashvili equation* (Kadomtsev and Petviashvili 1970). This equation is denoted as a two-dimensional version of the KdV equation with solutons in one direction only

$$(u_t - 6uu_x + u_{xxx})_x + 3u_{yy} = 0. \tag{14}$$

We could not find solitons, but a moving singularity through the ansatz $u^{(0)} = \log(kx + ly - \omega t)$. The iterates converged asymptotically

$$u^{(n)} = S^{(n)}k^2/(kx + ly - \omega t)^2 - \tfrac{1}{6}\omega/k + \tfrac{1}{2}l^2/k^2. \tag{15}$$

For example $S^{(5)} \approx 2.131$, $S^{(10)} \approx 1.982$ and $S^{(15)} \approx 2.002$, from which we derived the (correct) value $S = 2$.

Our last example, the *Liouville equation* (Liouville 1853), written for the characteristic variables x and t,

$$u_{xt} = e^u \tag{16}$$

is outstanding because we were able to find the exact solution. We used the iteration scheme $u^{(n+1)} = \log(u_{xt}^{(n)})$ and tried a few guesses for $u^{(0)}$ containing arbitrary functions $X = X(x)$ and $T = T(t)$. We found that $u^{(0)} = \log[X + T]$, converged to the exact solution

$$u(x, t) = \log \frac{2X'T'}{(X+T)^2} \tag{17}$$

within two steps. The solution (17) was derived by Liouville in 1853 (compare also the Bäcklund transform, e.g. Drazin 1983).

Let us summarize that, by a rather simple method, a few exact solutions of some nonlinear partial differential equations can be derived. We think that the method, which incorporates the common ansatz method, might be applied successfully to further cases, especially if some information about the solution is available.

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